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Reduced model for the scattering by small obstacles in time domain

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ABSTRACT

In the context of acoustic imaging, it is rather difficult to observe heterogeneities with characteristic length smaller than the wave length emitted by the scanner. However, it is possible to detect small heterogeneities in homogeneous media by using high performance computing. In this work, we will present a method, which has been developed in [1], to compute the field scattered by a small obstacle with low computation burden based on the matched asymptotic expansions.

Let us consider a small obstacle B_ε equipped with Dirichlet boundary conditions :

$$B_\varepsilon = \varepsilon \hat{B} = \Big\{(x,y,z) : \Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z}{\varepsilon}\Big) \in \hat{B}\Big\}, \tag{1}$$

with $\hat{B}$ a reference shape and $\partial B_\varepsilon = \varepsilon \partial \hat{B}$ its boundary. The propagation domain Ω_ε consists of the exterior to the obstacle B_ε :

$$\Omega_\varepsilon = \mathbb{R}^3 \backslash B_\varepsilon. \tag{2}$$

We denote by u_ε the solution of the 3 dimensional time-domain wave equation equipped with the Dirichlet boundary condition and initial conditions

$$\begin{cases} \frac{\partial^2 u_\varepsilon}{\partial t^2}(\mathbf{x},t) - c^2 \Delta u_\varepsilon(\mathbf{x},t) = 0, & \mathbf{x} \in \Omega_\varepsilon,\ t \geq 0, \\ u_\varepsilon(\mathbf{x},t) = 0, & \mathbf{x} \in \partial\Omega_\varepsilon,\ t > 0, \\ u_\varepsilon(\mathbf{x},0) = g_0(\mathbf{x}), \quad \partial_t u_\varepsilon(\mathbf{x},0) = 0 & \mathbf{x} \in \Omega_\varepsilon. \end{cases} \tag{3}$$

The matching of asymptotic expansions is an asymptotic domain decomposition method with overlapping. It consists in representing the solution with a far-field expansion far away from the obstacle and a near-field expansion near the obstacle. These two expansions are matched in a transition zone with the so-called Van Dyke matching conditions.

The far-field expansion is defined on the far-field domain $\Omega_* = \mathbb{R}^3 \setminus \{\mathbf{0}\}$ consisting of the limit of Ω_ε for ε varying to 0. It takes the form of a Taylor series :

$$u_{\varepsilon,I}\,(\mathbf{x},t) = \sum_{i=0}^{I} u_i(\mathbf{x},t)\varepsilon^i. \tag{4}$$

The first term of this expansion $u_0 : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is the limit of u_ε for ε varying to 0. It is a regular solution over all $\mathbb{R}^3$ of the time-domain wave equation :

$$\begin{cases} \dfrac{\partial^2 u_0}{\partial t^2}(\mathbf{x},t) - c^2 \Delta u_0(\mathbf{x},t) = 0, & \mathbf{x} \in \mathbb{R}^3,\ t > 0, \\ u_0(\mathbf{x},0) = g_0(\mathbf{x}), \quad \partial_t u_0(\mathbf{x},0) = 0, & \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

The next coefficients $u_i : \Omega_* \longrightarrow \mathbb{R}$ of this expansion are solutions of the homogeneous time-domain wave equation :

$$\begin{cases} \dfrac{\partial^2 u_i}{\partial t^2}(\mathbf{x},t) - c^2 \Delta u_i(\mathbf{x},t) = 0, & \mathbf{x} \in \Omega_*,\ t > 0, \\ u_i(\mathbf{x},0) = 0, \quad \partial_t u_i(\mathbf{x},0) = 0, & \mathbf{x} \in \Omega_*, \end{cases}$$

which are singular in the neighbourhood of $\mathbf{x} = 0$. This power series aims at approximating the solution u_ε at fixed $\mathbf{x} \neq \mathbf{0}$:

$$u_\varepsilon(\mathbf{x},t) - u_{\varepsilon,I}(\mathbf{x},t) = \underset{\varepsilon \to 0}{O}(\varepsilon^{I+1}). \tag{5}$$

In the case of a spherical obstacle, $B_\varepsilon = \{\mathbf{X} \in \mathbb{R}^3 : \|\mathbf{X}\| \leq \varepsilon\}$, the second order far-field expansion can be interpreted as the field generated by a point source. It takes the form

$$u_{\varepsilon,2}(\mathbf{x},t) = u_0(\mathbf{x},t) - \varepsilon \frac{u_0(0,t-t_0)}{r} - \varepsilon^2 \left(\frac{\partial_t u_0(0,t-t_0)}{r} \right) \quad \text{with } t_0 = \frac{r}{c}.$$

The near-field domain $\widehat{\Omega} = \frac{\Omega_\varepsilon}{\varepsilon}$ consists in the normalization of the original domain Ω_ε. The near-field expansion aims at approximating $U_\varepsilon(\mathbf{X},t) = u_\varepsilon(\varepsilon \mathbf{X},t)$ at fixed $\mathbf{X} \in \widehat{\Omega}$

$$\sum_{i=0}^{+\infty} U_i(\mathbf{X},t)\varepsilon^i. \tag{6}$$

The coefficients of the near field expansion satisfy the hierarchical Laplace equation

$$\begin{cases} \Delta U_i(\mathbf{X},t) = \dfrac{\partial_t^2 U_{i-2}(\mathbf{X},t)}{c^2}, \quad \mathbf{X} \in \widehat{\Omega},\ t > 0 \text{ with } U_i \equiv 0 \text{ for } i < 0, \\ U_i(\mathbf{X},t) = 0, \quad \mathbf{X} \in \partial\widehat{\Omega}, \end{cases} \tag{7}$$

Numerical results. We will present a comparison between a direct numerical simulation and the simulation of the reduced model. The direct numerical simulation requires to refine the mesh close to the small obstacle. This leads a high computation cost (small mesh step and small time step) and the DNS require high performance computing whereas the solution of the reduced model is easy to compute. The numerical simulations are achieved thanks to an interior penalty discontinuous galerkin method.

REFERENCES

[1] V. Mattesi, *Propagation des ondes dans un domaine comportant des petites hétérogénéités: modélisation asymptotique et calcul numérique*, PhD thesis, Université de Pau et des pays de l'Adour, 2014.