X-DMS 2015
eXtended Discretization MethodS
September 9-11, 2015, Ferrara, Italy

Multiscale superposition and profile computations for linear elasticity

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Key words: Multiscale expansions, artificial boundary conditions, problems in unbounded domains.

ABSTRACT

In many physical situations, one has to consider objects whose geometry involves different scales. Typically, to the macroscopic description should be added a microscopic level of details: this is the case for granulates inside concrete, or bumps on a shell. The questions we are interested in deal in particular with mechanical properties of such materials. The mathematical modeling of these situations consists usually of a system of partial differential equations posed in a (2D or 3D) domain standing for the real geometry. While the theoretical aspects are usually unaected by microscopic inhomogeneities, the numerics need special care. Indeed, taking into account two different scales in a finite element code requires an adapted mesh refinement in the vicinity of the defects. Resulting computations can become prohibitively costly. Hence, usually, only the macroscopic description is preserved. The in uence of local inhomogeneities on the global behavior of the material is then ignored. We aim at designing a numerical method involving the two geometric scales, with a reasonable computational cost. We consider the

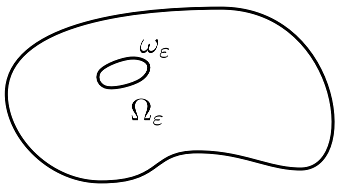

Figure 1: A simplified geometry.

Lamé equations in a domain $\Omega_\varepsilon = \Omega_0 \setminus \omega_\varepsilon$, where $\omega_\varepsilon = \varepsilon\omega$ is a small inclusion inside Ω_0 (we assume that the origin of the coordinates lies inside ω and Ω_0).

$$\begin{cases} -\mu\Delta\mathbf{u}_\varepsilon - (\lambda+\mu)\,\mathbf{grad}\;\mathrm{div}\,\mathbf{u}_\varepsilon = \mathbf{f} \text{ in } \Omega_\varepsilon, \\ \mathbf{u}_\varepsilon = 0 \text{ on } \partial\Omega_0, \\ \sigma\cdot\mathbf{n} = 0 \text{ on } \partial\omega_\varepsilon, \end{cases} \tag{1}$$

where λ and μ are the Lamé coefficients, and $\mathbf{f}$ a given loading. It can be shown – see [3, 2] – The expansion of $\mathbf{u}_\varepsilon$ takes the form

$$\mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) - \varepsilon\,\mathbf{a}\cdot\mathbf{V}\left(\frac{\mathbf{x}}{\varepsilon}\right) + \mathbf{r}_\varepsilon(\mathbf{x}), \tag{2}$$

with

- $\mathbf{u}_0$ solve Problem (1) for $\varepsilon = 0$ (i.e. solution in the unperturbed domain Ω_0),
- the vectorial coefficient $\mathbf{a}$ depends on the the stress values associated with $\mathbf{u}_0$ at point $\mathbf{0}$.
- the *profile* $\mathbf{V}$ is solution to an exterior problem

$$\begin{cases} -\mu\Delta\mathbf{V} - (\lambda+\mu)\,\mathbf{grad}\,\mathrm{div}\,\mathbf{V} = \mathbf{0} \text{ in } \mathbb{R}^2 \setminus \omega, \\ \sum_{j=1}^{2} \sigma_{ij}(\mathbf{V})\mathbf{n}_j = \mathbf{G}_i \text{ on } \partial\omega, \end{cases} \tag{3}$$

  where the datum $\mathbf{G}$ only depend on the geometry of ω.

- the remainder $\mathbf{r}_\varepsilon$ satisfies the following estimate in the energy norm

$$\|\mathbf{r}_\varepsilon\|_{\mathrm{H}^1(\Omega_\varepsilon)} \leq C\varepsilon^2. \tag{4}$$

Solving Problem (3) is the crucial point of the method. A classical approach consists in bounding the computational domain with a Ball $\mathcal{B}_R$ of radius R, and setting an approximate boundary condition on $\partial\mathcal{B}_R$. We have shown in [1] that a condition at order 1 reads

$$\frac{R(1+\nu)}{E}\sigma(\mathbf{u})\cdot\mathbf{n} + \frac{1}{2}\begin{bmatrix} -\frac{\nu}{2(1-\nu)} & 0 \\ 0 & \frac{1-\nu}{1-2\nu} \end{bmatrix} \Delta_\tau \mathbf{u} + \mathbf{u} = \mathbf{0}, \tag{5}$$

where Δ_τ denotes the Laplace-Beltrami operator on $\partial\mathcal{B}_R$. Existence and uniqueness do not follow from standard techniques for this non-coercive problem. The general situation is that, except for a countable set of values for R, the problem with condition (5) admits a unique solution. It is actually shown only for ω near a ball, but numerical computations suggest it remains true in a wider framework.

This work has been supported by the french ANR projects MACADAM (JCJC06_139561) and ARAMIS (12 BS01-0021).

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