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EXTENDED HYBRIDIZABLE DISCONTINUOUS GALERKIN (X-HDG) FOR VOID PROBLEMS ♣

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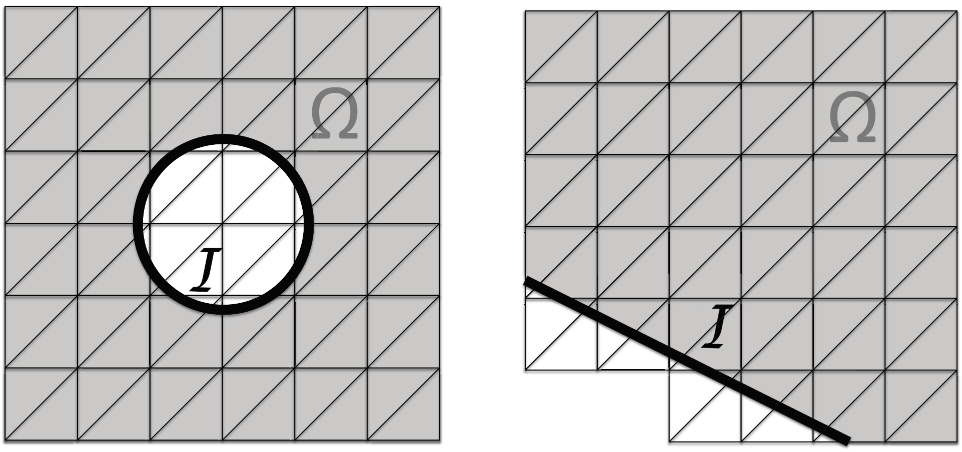

Figure 1. Two examples of domain with a void: a circular void boundary and a straight interface, *I* in black. The mesh covers the domain (in grey) and fits only the exterior boundary.

Even though the Hybridizable Discontinuous Method (HDG) is a novel method proposed just a few years ago [1], it has nowadays been successfully applied to all kind of problems, specially in the field of Computational Fluid Dynamics (CFD); see, for instance [2] for its application to the Navier--Stokes equations, or [3] for an efficiency study in front of Continuous Finite Elements (CFE) in the context of wave problems.
HDG inherits all the advantages of high-order Discontinous Galerkin (DG) methods that have made them so popular in CFD in the last decade, such as local conservation of quantities of interest, intrinsic stabilization thanks to a proper definition of numerical fluxes at element boundaries, suitability for code vectorization and parallel computation, and suitability for adaptivity. But, HDG outperforms other DG methods for problems involving self-adjoint operators, due to two main peculiarities: hybridization and superconvergence properties. The hybridization process drastically reduces the number of degrees of freedom in the discrete problem, similarly to static condensation in the context of high-order CFE. For instance, in a Laplace equation the unknowns reduce to the approximation of the trace of the solution at the mesh skeleton, i.e. the sides (or faces in 3D) of the mesh; and in incompressible flow problems, the final unknowns correspond to just the trace of the velocity at the mesh skeleton plus one scalar representing the mean of the pressure at every element. On other hand, HDG is based on a

mixed formulation that, differently to CFE or other DG methods, is stable even when all variables (primal unknowns and derivatives) are approximated with polynomials of the same degree k. Consequently, convergence of order k+1 in L2 norm is proved not only for the primal unknown, but also for its derivatives. Therefore, a simple element-by-element postprocess of the derivatives leads to a superconvergent approximation of the primal variables, with convergence of order k+2 in L2 norm. The superconvergent solution can also be used to compute an efficient error estimator and define an adaptivity procedure [2].
However, despite the interest in the development and application of HDG during the last years, there is still work to be done for the efficient solution of problems with moving boundaries and interfaces. A methodology for the solution of elliptic problems with meshes not fitting the boundary is proposed in [4]. The solution at the boundary is extrapolated from nodal values of the computational mesh; consequently, some restrictive requirements on the distance from the computational mesh to the boundary are necessary to achieve optimal convergence, limiting the practical applicability of the proposal.
An alternative strategy for the HDG solution of interface problems, based on an eXtended Finite Element (X-FEM) philosophy is proposed here: the eXtended Hybridizable Discontinous Galerkin (X-HDG). X-HDG inherits the advantages of X-FEM methods (the computational mesh is not required to fit the interface, simplifying and reducing the cost of mesh generation and, in particular, avoiding continuous remeshing for evolving interfaces or boundaries), while keeping the computational efficiency, stability, accuracy and optimal convergence of HDG. Differently to [4], here the computational mesh always covers the domain and, therefore, no extrapolations are required, leading to a more robust method.
The local problem at elements not cut by the interface, and the global problem, are discretized as usual in HDG. At every cut element, an auxiliary trace variable on the boundary is introduced, which is eliminated afterwards using the boundary conditions on the interface, keeping the original unknowns.
A robust and efficient methodology for numerical integration in cut elements is also proposed. Similarly to [5], a k-th degree parametrization for the approximation of the interface in each cut element is considered. However, here the parametrization may be piecewise polynomial, getting rid of the mesh requirements in [5], and being capable of handling more complicated situations that may appear in high-order computations, such as bubbles inside an element, or an element divided in more than two regions by the interface.

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