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Partition of unity methods for approximation of point water sources in porous media

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ABSTRACT

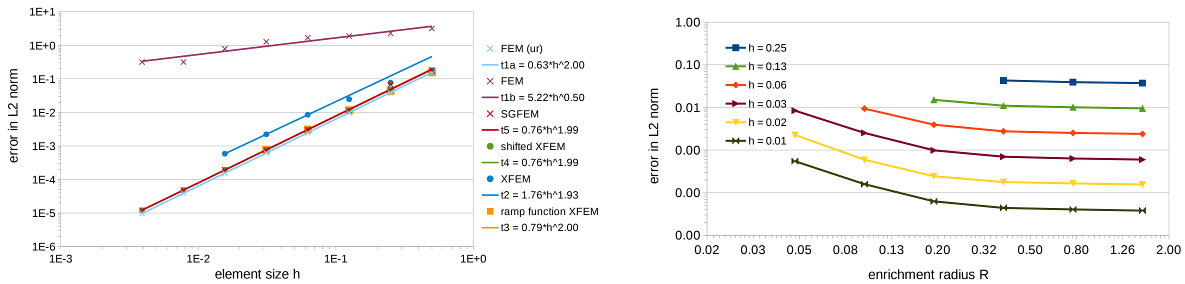
Introduction. A steady water flow problem in an 2D aquifer-well system is solved using several partition of unity (PU) methods. Standard FEM approximation is unable to approximate singular beaviour of pressure in the vicinity of wells, unless the elements of a mesh are smaller than the wells. In order to improve the approximation, the standard finite element space is enriched with an analytical solution to a Laplace problem with a point source on the whole $\mathbf{R}^2$ space. The enrichment is incorporated into the solution via PU methods. Convergence in terms of L^2 norm of the error against analytical solution is measured and the optimal order of convergence rate is demonstrated for some of the PU methods. Next, the error of adaptive integration is analysed and a new adaptive strategy is proposed. The influence of the choice of the enriched domain is investigated and its impact on the error is demonstrated numerically.

PU methods comparison. We are comparing four different PU methods: standard XFEM (extended finite element method), XFEM with a ramp function (according to R. Gracie and J. R. Craig [5, 2]), XFEM with both a ramp function and a shifting (denoted as corrected XFEM by T. P. Fries in [3, 4]), SGFEM (stable generalized FEM) developed by Babuška and Banerjee [1].

In case of XFEM, a ramp function can be used to blend enriched and unenriched part of the solution on the interface of a local enrichment area (blending elements). We show the higher error on the blending elements in case of standard XFEM and ill-conditioning of the system matrix when using the ramp function. The corrected XFEM does not suffer from either of those in our model.

The SGFEM uses the difference between the enrichment function and its finite element interpolation in the enrichment term. We achieve the same error and optimal convergence rate 2.0 with SGFEM as with the corrected XFEM. The advantage of SGFEM is avoiding the special treatment of blending elements with ramp function and thus a lower count of enriched degrees of freedom in comparison to corrected XFEM. See the convergence graph of PU methods in comparison to standard FEM with bilinear finite elements in figure 1a.

Adaptive integration. The adaptive integration of non-polynomial shape functions presented in [5, 2] was found insufficient, especially when an element edge approached a well. We have



(a) Convergence graph displaying dependence of the error in L^2 norm on the element size. Standard FEM is compared with several types of PU methods.

(b) Dependence of the error in L^2 norm on the enrichment radius for different element sizes.

made an asymptotic analysis of the integration error and suggest new rules. Only then, we managed to obtain optimal convergence rates (figure 1a) independently of the well position.

The choice of enrichment area (set of enriched elements) is not trivial, as we do not want to enrich elements redundantly. In case of the singularity, the area is radial and can be defined by a single parameter – enrichment radius. Assuming that the solution can be split into the singular (enriched solution part) and the regular part (standard FEM solution part) $u = u_s + u_r$ and using standard error estimate for elliptic PDE, we estimate the enrichment radius, such that the error of u_s on the unenriched area is not higher than the error of regular part u_r. According to our estimate, the optimal enrichment radius is 0.36 in our model which roughly matches a breakpoint in the plots in figure 1b.

Conclusion. We compared several PU methods on a 2D steady flow model containing a point singularity. We suggested a new adaptive integration rules which yield in optimal convergence rate. We estimated optimal enrichment area for our model and validated it with numerical experiment. In future, we would like to use our experience to apply PU methods in mixed hybrid formulation to better approximate flow in fractured porous media.

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