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A 3D-1D coupled model for elastic response of fibre-reinforced composites

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ABSTRACT

We propose a mathematical model for the numerical study of mechanical properties of fibre-reinforced frames used in car industry. The frame typically consists of a hollow plastic tube (matrix) winded up by one or several unidirectional bundles of carbon fibres which are sealed to the matrix. The finite element (FE) analysis of full 3D models has several limitations: One has to use either complicated and large meshes to resolve the complex fibre structures or average the material properties of matrix and fibres e.g. by the Voigt law of mixtures.

Our approach consists in treating the matrix and fibre domains separately, which allows the use of non-compatible meshes and possibly different material laws. In addition we introduce a coupling mechanism between the displacements in matrix and fibres. We consider the matrix domain $\Omega \subset \mathbb{R}^3$ and the one-dimensional fibre manifold $\Sigma \subset \Omega$ and look for the displacements $\boldsymbol{u}_m : \Omega \to \mathbb{R}^3$, $\boldsymbol{u}_f : \Sigma \to \mathbb{R}^3$. For simplicity we consider the linear elasticity

$$-\operatorname{div} \boldsymbol{\sigma}_m(\boldsymbol{u}_m) = \boldsymbol{f}_m \text{ in } \Omega, \quad -\operatorname{div} \boldsymbol{\sigma}_f(\boldsymbol{u}_f) = \boldsymbol{f}_f \text{ on } \Sigma \tag{1}$$

together with appropriate boundary conditions. The fibres are assumed to be tightly connected to the matrix, hence we require

$$\boldsymbol{u}_m = \boldsymbol{u}_f \text{ on } \Sigma. \tag{2}$$

The stress tensor $\boldsymbol{\sigma}_m$ represents linear isotropic material. On the fibres we consider a simplified linear spring model or the beam equations based on the Euler-Bernoulli or the Timoshenko theory [2].

Problem (1)–(2) is approximated by the finite element method. We do not impose any assumptions on the compatibility of the 1D and 3D computational mesh. The equality of displacements is enforced by the constraint

$$\int_E (\boldsymbol{u}_m - \boldsymbol{u}_f)\, dx = 0 \tag{3}$$

on each element E of the 1D mesh. This formulation leads to a saddle-point algebraic problem:

$$\begin{bmatrix} \mathbb{K}_m & 0 & \mathbb{C}_m^\top \\ 0 & \mathbb{K}_f & \mathbb{C}_f^\top \\ \mathbb{C}_m & \mathbb{C}_f & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_m \\ \boldsymbol{U}_f \\ \boldsymbol{\Lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_m \\ \boldsymbol{F}_f \\ 0 \end{bmatrix}.$$

Here $\mathbb{K}_m$, $\mathbb{K}_f$ are the stiffness matrices, $\mathbb{C}_m$, $\mathbb{C}_f$ are matrices realizing the constraints (3), $\boldsymbol{U}_m$, $\boldsymbol{U}_f$ are the vectors of degrees of freedom for displacements and $\boldsymbol{\Lambda}$ is the vector of Lagrange multipliers.

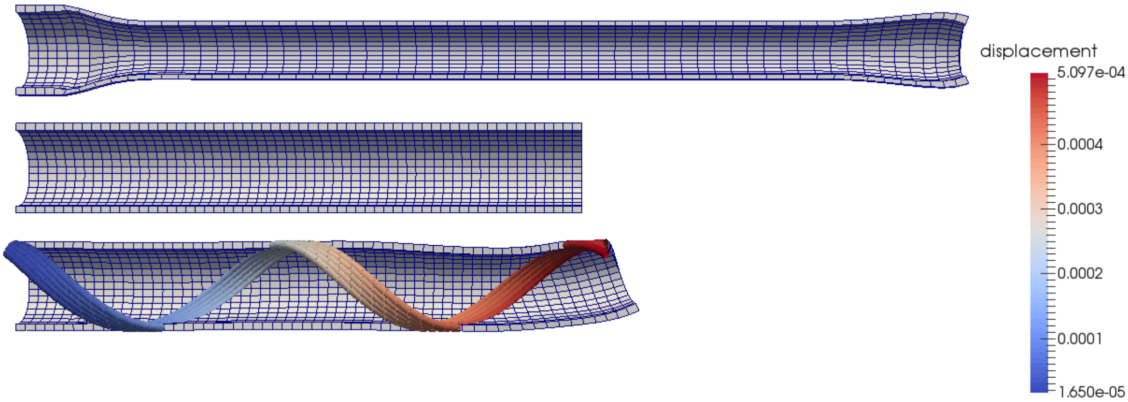

Figure 1: Sample stretched along the axis. Top: plain matrix, middle: reference configuration, bottom: reinforced.

NUMERICAL EXAMPLES

The method has been implemented using the deal.II FE library [1]. The 3D displacements are discretized by bilinear hexahedral elements, for displacements on fibres we use biquadratic line elements. The Lagrange multipliers are constant on 1D elements. In the presented examples we used the elastic moduli for polyurethane matrix and carbon fibres. In Figure 1 we demonstrate the effect of reinforcement by a single fibre bundle on a stretched composite sample. The response of a sample reinforced by one or two bundles to bending force is shown in Figure 2.

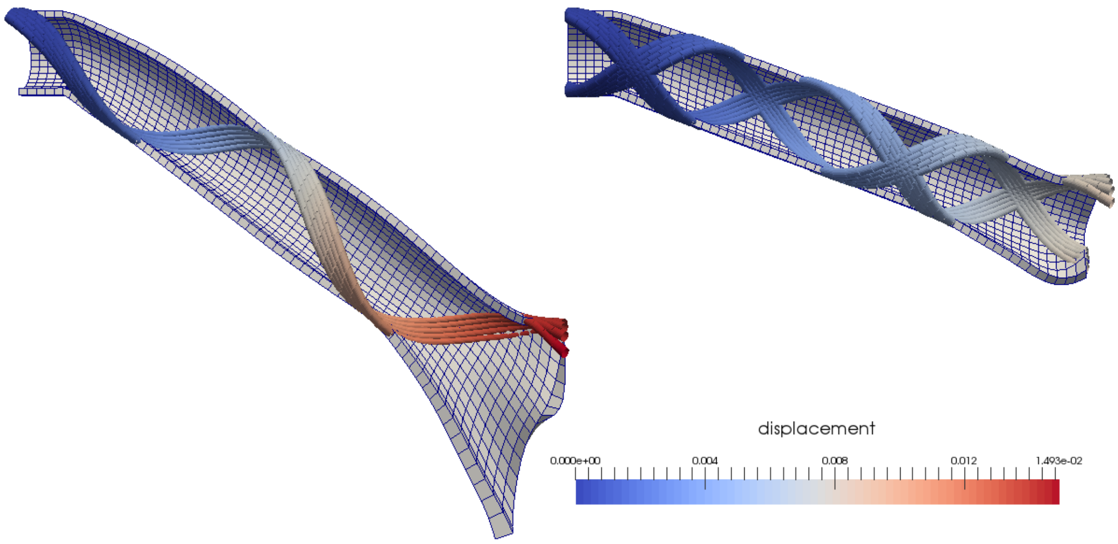

Figure 2: Bending test. Left: one bundle of fibres, right: two bundles.

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