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Low-order reconstruction operators on polyhedral meshes

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ABSTRACT

Reconstruction (or lifting) operators, mapping degrees of freedom (DoFs) to functions living in a finite-dimensional space, play a salient role in the numerical approximation of partial differential equations (PDEs). In compatible or mimetic discretization schemes (see [10] and references therein), DoFs are defined as the codomain of de Rham (or reduction) operators and are attached to some geometric entities of an underlying three-dimensional mesh (e.g. vertices, edges, faces, and cells) according to the physical nature of fields. In what follows, a space of DoFs is generically denoted by $\mathcal{X}$ and the reconstruction and de Rham operators associated with this space are respectively denoted by $\mathtt{L}_{\mathcal{X}}$ and $\mathsf{R}_{\mathcal{X}}$. Loosely speaking, $\mathtt{L}_{\mathcal{X}}$ provides an approximation of a right inverse of $\mathsf{R}_{\mathcal{X}}$. The reconstruction operator is said to be of low-order when the composition $\mathtt{L}_{\mathcal{X}} \cdot \mathsf{R}_{\mathcal{X}}$ (yielding an interpolation operator) leaves cell-wise constant fields invariant.

Our main focus here is the generic design of reconstruction operators on polyhedral meshes in order to build discrete Hodge operators (or Hodge inner products). The discrete Hodge operator is the cornerstone of many compatible discretizations [1, 12, 10] and is related to reconstruction operators through the following identity:

$$\mathsf{H}_{\alpha}^{\mathcal{X}}(\mathbf{a}_1, \mathbf{a}_2) := \int_{\Omega} \mathtt{L}_{\mathcal{X}}(\mathbf{a}_1) \cdot \alpha \cdot \mathtt{L}_{\mathcal{X}}(\mathbf{a}_2), \qquad \forall \mathbf{a}_1, \mathbf{a}_2 \in \mathcal{X},$$

where $\Omega \subset \mathbb{R}^3$ is the computational domain discretized by the polyhedral mesh and α is a phenomenological parameter such as a conductivity. In the context of Finite Elements, Whitney reconstruction functions provide a classical example of reconstruction operators on tetrahedral meshes. On polyhedral meshes, generic design principles of reconstruction operators have been proposed in [3, 4, 11]. These operators are built in each mesh cell so that suitable matching conditions are satisfied at mesh interfaces to ensure the conformity of the reconstruction (in the sense that the operator maps to the appropriate Sobolev space).

Relying on the Compatible Discrete Operator (CDO) framework [1], we adopt an alternative viewpoint which unifies reconstruction operators devised in [5, 8, 9]. In this context, the reconstruction operators typically map onto piecewise constant functions on a submesh (thereby discarding local conformity), while their composition with the de Rham operator remains single-valued. Another contribution is that we identify the design principles that reconstruction operators have to verify to ensure the convergence of the numerical scheme for elliptic problems.

Specifically, we establish the equivalence between a one-level design strategy of the reconstruction operator as considered in [2, 5] and a two-level design strategy as considered in [3, 9, 6]. This second strategy, which decomposes the reconstruction operator into the sum of a consistent part and a stabilization part, provides a systematic construction principle where the only user-dependent design lies in the stabilization part, while the consistent part is fixed.

As an illustration, we present piecewise constant reconstruction operators in each mesh cell for all types of DoFs. In the case of edge-based reconstruction operators, we study the impact of the stabilization parameter in terms of accuracy and computational cost for the numerical approximation of anisotropic diffusion problems on several polyhedral meshes. Under- and over-penalized values of the stabilization parameter have a negative impact on the preservation of bounds and on accuracy and costs. For the problem considered, appropriate choices are values closed to those proposed in [5, 9]. These conclusions are to be confirmed by further numerical tests.

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