X-DMS 2015
eXtended Discretization MethodS
September 9-11, 2015, Ferrara, Italy

A discontinuous-skeletal method for advection-diffusion-reaction on general meshes

Daniele A. Di Pietro[1], **Jérôme Droniou**[2] **and Alexandre Ern**[3]
[1] University of Montpellier, I3M, 34057 Montpellier CEDEX 5, France
[2] School of Mathematical Sciences, Monash University, Victoria 3800, Australia
[3] University Paris-Est, CERMICS (ENPC), 6–8 avenue Blaise Pascal, 77455 Marne-la-Vallée, France

Key words: Discontinuous-skeletal methods, general meshes, degenerate advection-diffusion-reaction.

ABSTRACT

We present an approximation method for degenerate advection-diffusion-reaction equations where the (generalized) degrees of freedom (DOFs) after static condensation are polynomials of order $k \geqslant 0$ at mesh faces. Since such faces constitute the mesh skeleton, and since DOFs can be chosen independently at each face, we use the terminology discontinuous-skeletal method. The proposed method has various assets: (i) Fairly general meshes, with polytopal and nonmatching cells, are supported; (ii) Arbitrary polynomial orders, including the case $k = 0$, can be considered; (iii) The error analysis covers the full range of Péclet numbers; (iv) Computational costs remain moderate since skeletal DOFs lead to a compact stencil.

The problem is defined as follows. Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$, be an open bounded connected polytope of boundary $\partial\Omega$ and unit outer normal $\boldsymbol{n}$. The diffusion coefficient $\nu : \Omega \to \mathbb{R}^+$ is assumed to be piecewise constant on a partition $P_\Omega := \{\Omega_i\}_{1\leqslant i\leqslant N_\Omega}$ of Ω into polytopes and such that $\nu \geqslant \underline{\nu} \geqslant 0$ almost everywhere in Ω. More general diffusion coefficients can be treated following the ideas of [4]. For the advective velocity $\boldsymbol{\beta} : \Omega \to \mathbb{R}^d$, we assume that $\boldsymbol{\beta} \in \mathrm{Lip}(\Omega)^d$ and, for the sake of simplicity, that $\boldsymbol{\nabla}\cdot\boldsymbol{\beta} \equiv 0$. For the reaction coefficient $\mu : \Omega \to \mathbb{R}$, we assume and that μ is bounded from below by a real number $\mu_0 > 0$. We introduce the following sets (cf. Figure 1):

$$\Gamma_{\nu,\boldsymbol{\beta}} := \{\boldsymbol{x} \in \partial\Omega \mid \nu > 0 \text{ or } \boldsymbol{\beta}\cdot\boldsymbol{n} < 0\}, \qquad \mathcal{I}^{\pm}_{\nu,\boldsymbol{\beta}} := \{\boldsymbol{x} \in \mathcal{I}_\nu \mid \pm(\boldsymbol{\beta}\cdot\boldsymbol{n}_I)(\boldsymbol{x}) > 0\},$$

where $\mathcal{I}_\nu$ is the diffusive/nondiffusive interface and $\boldsymbol{n}_I$ is the unit normal to $\mathcal{I}_\nu$ pointing out of the diffusive region. More precisely, $\mathcal{I}_\nu$ is the set of points in Ω located at an interface between two distinct subdomains Ω_i and Ω_j of P_Ω such that $\nu_{|\Omega_i} > \nu_{|\Omega_j} = 0$. We assume that $(\boldsymbol{\beta}\cdot\boldsymbol{n}_I)(\boldsymbol{x}) \neq 0$ for a.e. $\boldsymbol{x} \in \mathcal{I}_\nu$. For given source term $f \in L^2(\Omega)$ and boundary datum $g \in L^2(\Gamma_{\nu,\boldsymbol{\beta}})$, the problem reads

$$\begin{aligned} \boldsymbol{\nabla}\cdot(-\nu\boldsymbol{\nabla}u + \boldsymbol{\beta}u) + \mu u &= f && \text{in } \Omega\backslash\mathcal{I}_\nu, \\ [\![-\nu\boldsymbol{\nabla}u + \boldsymbol{\beta}u]\!]\cdot\boldsymbol{n}_I &= 0 && \text{on } \mathcal{I}_\nu, \\ [\![u]\!] &= 0 && \text{on } \mathcal{I}^+_{\nu,\boldsymbol{\beta}}, \\ u &= g && \text{on } \Gamma_{\nu,\boldsymbol{\beta}}, \end{aligned}$$

where $[\![\cdot]\!]$ denotes the jump across $\mathcal{I}_\nu$ (the sign is irrelevant). Notice that the boundary condition is enforced at portions of the boundary touching a diffusive region or a nondiffusive region provided the advective field flows into the domain. A weak formulation for this problem has been

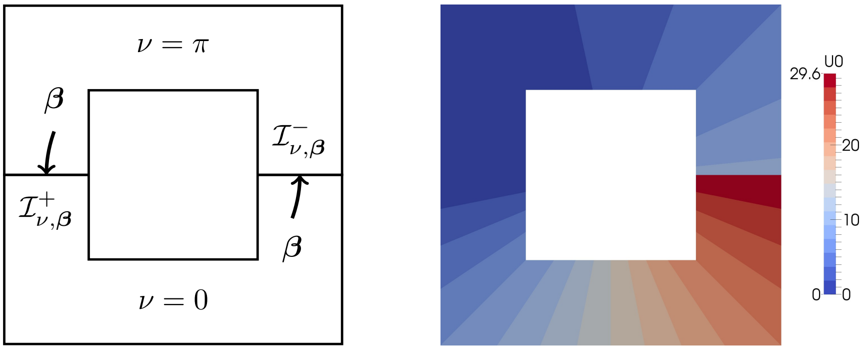

Figure 1: Configuration of a degenerate advection-diffusion-reaction problem (left) and exact solution (right). The jump discontinuity across $\mathcal{I}^-_{\nu,\beta}$ is clearly visible.

analyzed in [1] (in the non-degenerate case $\underline{\nu} > 0$, $\Gamma_{\nu,\beta} = \partial\Omega$ and the usual weak formulation in the space $H^1_0(\Omega)$ holds).

The starting point for the present discontinuous-skeletal method is the Hybrid-High Order (HHO) method designed in [3]. Its extension to advection-diffusion-reaction equations entails several new ideas: (i) We devise a local reconstruction of the advective derivative from cell- and face-based DOFs using an integration by parts formula; (ii) Stability for the advective contribution is ensured by terms that penalize the difference between cell- and face-based DOFs at faces, and which therefore do not preclude the possibility of performing static condensation and do not enlarge the stencil; as in [2], the stability terms are formulated in a rather general form so as to include various approaches used in the literature, e.g., upwind, locally θ-upwind, and Scharfetter–Gummel schemes; (iii) Boundary conditions are enforced weakly so as to achieve robustness in the full range of Péclet numbers. An additional novel feature of the present work is that our analysis also includes the case of locally degenerate advection-diffusion-reaction equations, where the diffusion coefficient vanishes on a (strict) subset of the computational domain. Such problems are particularly delicate from a numerical viewpoint since, as pointed out, the numerical method has to capture the jumps of the exact solution at the diffusive/nondiffusive interface.

REFERENCES

[1] D. A. Di Pietro, A. Ern, and J.-L. Guermond. Discontinuous Galerkin methods for anisotropic semi-definite diffusion with advection. *SIAM J. Numer. Anal.*, 46(2):805–831, 2008.

[2] L. Beirão da Veiga, J. Droniou, and M. Manzini. A unified approach for handling convection terms in finite volumes and mimetic discretization methods for elliptic problems. *IMA J. Numer. Anal.*, 31(4):1357–1401, 2010.

[3] D. A. Di Pietro, A. Ern, and S. Lemaire. An arbitrary-order and compact-stencil discretization of diffusion on general meshes based on local reconstruction operators. *Comput. Meth. Appl. Math.*, 14(4):461–472, 2014.

[4] D. A. Di Pietro and A. Ern. Hybrid high-order methods for variable-diffusion problems on general meshes. *C. R. Acad. Sci Paris, Ser. I*, 353:31–34, 2015.